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12

TERMINOLOGY

amplitude
equilibrium
general solution
period
periodic functions
periodic motion
phase shift
reciprocal functions
simple harmonic motion
sum and difference formulas
transformation
translation

Prior learning

TRIGONOMETRY

TRIGONOMETRIC FUNCTIONS AND GRAPHS

THE BASIC TRIGONOMETRIC FUNCTIONS

- find all solutions of $f(a(x - b)) = c$ where f is one of sin, cos or tan (ACMSM042)
- graph functions with rules of the form $y = f(a(x - b))$ where f is one of sin, cos, or tan. (ACMSM043)

THE RECIPROCAL TRIGONOMETRIC FUNCTIONS, SECANT, COSECANT AND COTANGENT

- define the reciprocal trigonometric functions; sketch their graphs and graph simple transformations of them. (ACMSM045)

TRIGONOMETRIC IDENTITIES

- convert sums $a \cos x + b \sin x$ to $R \cos(x \pm \alpha)$ or $R \sin(x \pm \alpha)$ and apply these to sketch graphs, solve equations of the form $a \cos x + b \sin x = c$ and solve problems (ACMSM048)

APPLICATIONS OF TRIGONOMETRIC FUNCTIONS TO MODEL PERIODIC PHENOMENA

- model periodic motion using sine and cosine functions and understand the relevance of the period and amplitude of these functions in the model. (ACMSM050) AC

12.01 PERIOD, AMPLITUDE AND PHASE SHIFT

You already know the shapes of the basic trigonometric functions $y = \sin(x)$, $y = \cos(x)$ and $y = \tan(x)$ from earlier work. You also saw how the constants a, b, c and d in $y = a\sin(x)$, $y = \sin(bx)$, $y = \sin(x + c)$ and $y = \sin(x) + d$ change the shape of the graph of $y = \sin(x)$. You have also done the same thing in Maths Methods for $y = \cos(x)$ and $y = \tan(x)$.

IMPORTANT

The **amplitude** of the functions $y = a\sin(x)$ or $y = a\cos(x)$ is a. It is the distance of the peaks and troughs from the average value of the function.

The **period** of a function $y = f(x)$ is the smallest value P such that $f(x) = f(x + P)$ for all x. The period of $y = \sin(bx)$ and $y = \cos(bx)$ is $\frac{2\pi}{b}$. The period of $y = \tan(bx)$ is $\frac{\pi}{b}$.

The **phase shift** of $y = \sin(x + c)$, $y = \cos(x + c)$ or $y = \tan(x + c)$ is c. It is the horizontal translation of the graphs from those of $y = \sin(x)$, $y = \cos(x)$ or $y = \tan(x)$. For positive values of c it is a displacement to the left, and for negative values of c it is to the right.

The value of d in $y = \sin(x) + d$ or $y = \cos(x) + d$ is the **vertical translation** of the graphs compared to those of $y = \sin(x)$ or $y = \cos(x)$. For positive values of d it is up, while for negative values it is down.

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Example 1

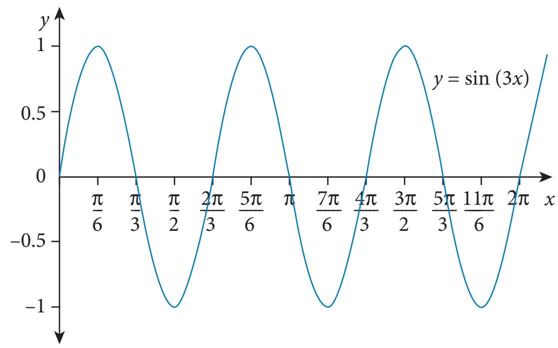
Sketch the graph of $y = \sin(3x)$ for $0 \le x \le 2\pi$.

Solution

Find the period of $y = \sin(3x)$ using $\frac{2\pi}{b}$.

Period $= \frac{2\pi}{3}$

The graph will repeat 3 times over the interval $0 \le x \le 2\pi$, with one cycle every $\frac{2\pi}{3}$ radians. It is compressed horizontally compared to $y = \sin(x)$.



The graphs of $y = -2\sin(x)$ and $y = \frac{1}{3}\cos(x)$ have the amplitudes 2 and $\frac{1}{3}$ respectively.

Example 2

Find the amplitude and period of the graph of $y = 4\cos\left(\frac{1}{2}x\right)$ and sketch its graph for one cycle.

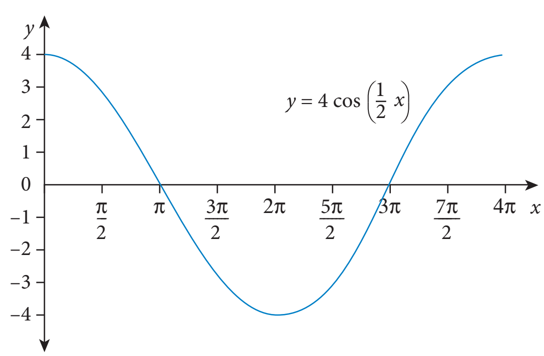
Solution

State the amplitude and period.

Amplitude $= 4$

Period $= \frac{2\pi}{\frac{1}{2}} = 4\pi$

Sketch the graph for one cycle, with the graphed stretched vertically by a factor of 4 compared to $\cos(x)$ and stretched horizontally by a factor of $\frac{1}{\frac{1}{2}} = 2$.



Example 3

Find the amplitude, period and phase shift of the graph of $y = 6\cos\left(x + \frac{\pi}{4}\right)$ and hence sketch its graph for one cycle.

Solution

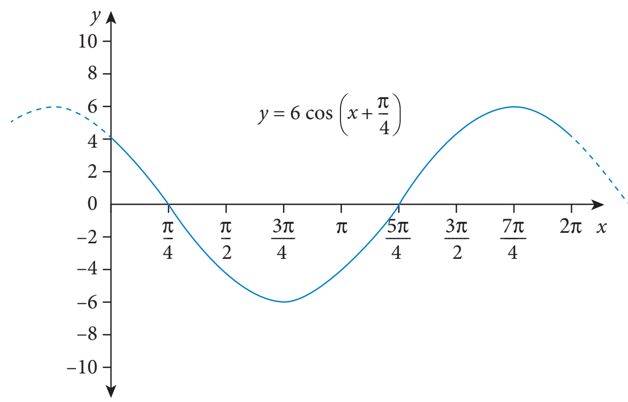
State the amplitude, period and phase shift.

Amplitude = 6

$\text{Period} = \frac{2\pi}{1} = 2\pi$

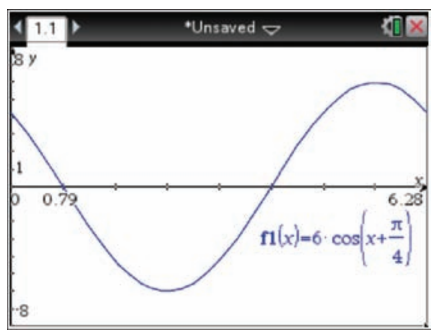
Phase change is $\frac{\pi}{4}$ units to the left.

The graph has the same period as $y = \cos(x)$ but it is shifted $\frac{\pi}{4}$ to the left so that everything occurs sooner. It is stretched vertically by a factor of 6 compared to $\cos(x)$.



TI-Nspire CAS

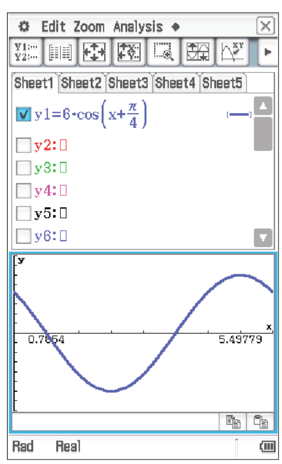
Make sure your calculator is set to radians, including the Graphing Angle in 9: Settings. Make the Window settings $0 \le x \le 2\pi$ with scale $\frac{\pi}{4}$ and $-8 \le y \le 8$ with scale 1.



ClassPad

Use the Graph application. Make sure your calculator is set to radians (**Rad**). Enter the function in $y1$ and tap the box to turn it on.

Tap [graph icon] to draw the graph. Set the **View Window** to $0 \le x \le 2\pi$ with scale $\frac{\pi}{4}$ and $-8 \le y \le 8$ with scale 1.



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EXERCISE 12.01 Period, amplitude and phase shift

Concepts and techniques

1 Examples 1, 2 Sketch the graphs of the following from $0 \le x \le 2\pi$ and check with a CAS calculator.

a $y = \cos(2x)$ b $y = \tan(2x)$ c $y = -3\sin(x)$ d $y = \frac{1}{2}\sin\left(\frac{2x}{3}\right)$

2 For $-\pi \le x \le \pi$, sketch a graph of each of the following and check with a CAS calculator.

a $y = \sin(2x)$ b $y = \sin\left(\frac{1}{2}x\right)$ c $y = \cos(3x)$ d $y = \sin(2x) + 3$

3 Example 4 For $0 \le x \le 2\pi$, sketch a graph of each of the following and check with a CAS calculator.

a $y = 2\sin(x)$ b $y = \frac{1}{2}\sin(x)$ c $y = 3\cos(x)$ d $y = 2\sin(x) + 1$

4 Sketch a graph of each of the following and check with a CAS calculator, showing a full cycle.

a $y = 3\sin(4x)$ b $y = \frac{1}{2}\cos(3x)$ c $y = 5\sin(2x)$ d $y = \frac{1}{3}\sin(8x) + 4$

5 Find equations of two of the vertical asymptotes and hence sketch the graph of the following for $0 \le x \le \pi$.

a $y = \tan(2x)$ b $y = 2\tan(3x)$ c $y = -\tan(x + \pi)$

6 Example 3 Sketch graphs of each of the following and check with a CAS calculator, showing a full cycle of each.

a $y = \sin\left(x - \frac{\pi}{2}\right)$ b $y = \sin\left(x + \frac{\pi}{2}\right)$ c $y = \cos\left(x - \frac{\pi}{6}\right)$ d $y = \cos\left(x + \frac{\pi}{6}\right)$

e $y = 2\sin\left(x - \frac{\pi}{3}\right)$ f $y = 2\sin\left(x + \frac{\pi}{3}\right)$ g $y = 3\cos\left(x - \frac{\pi}{4}\right)$ h $y = 3\cos\left(x + \frac{\pi}{4}\right)$

Reasoning and communication

7 A large natural harbour has an entrance that is only 900 m wide. The movement of tides keeps the entrance clear, but a channel has to be dredged to the port 14 km from the harbour entrance. At the port, the depth of water in the channel is given by $d = 4.7 + (h - m)\cos(0.5t - 0.21)$, where t is the time in hours after high tide at the harbour entrance, h is the height of the high tide and m is the height of the following low tide. On a particular day, the high tide is given as 2.9 m at 1:37 p.m. and the low tide as 0.5 m.

a Sketch a graph showing the depth of water in the channel from 12 noon on that day until 12 noon on the next day, assuming the model is valid for that time.

b When was the first low tide at the port on that day?

c For what period was the water at a depth of 5 m or more in the channel between successive low tides?

d How long was it between high tides?

e How long was it between high tide at the harbour entrance and high tide at the port?

8 The sales, S, in 100s of units, of a seasonal product are modelled by $S = 54.8 + 32.5\cos\left(\frac{\pi t}{6}\right)$ where t is the time in months ($t = 1$ is January and $t = 12$ is December).

a Draw a graph of the sales for a period of 12 months.

b Use the graph to determine the months for which sales exceed 6800 units.

12.02 GENERAL TRIGONOMETRIC FUNCTIONS

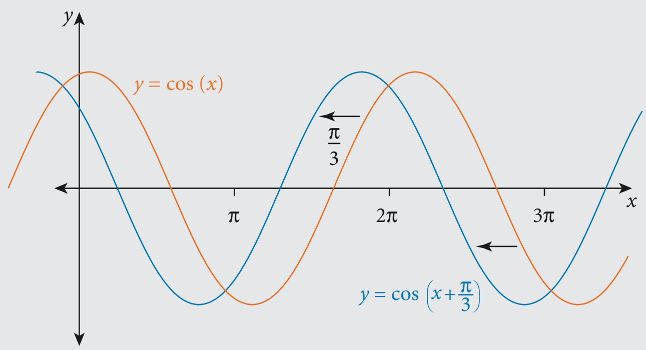
You have already sketched graphs of the form $y = f(bx + c)$ where f is one of sine, cosine, or tangent.

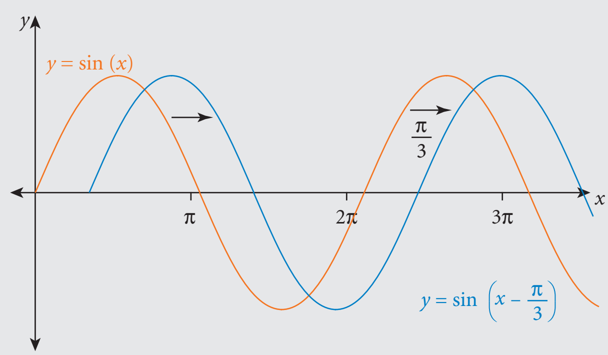
$y = f(bx + c)$ can also be expressed as $y = f\left(b\left(x + \frac{c}{b}\right)\right)$.

If you compare $y = \sin(bx + c)$ to $y = f(x + c)$ you can see that the period must be taken into account when finding a phase shift.

IMPORTANT

The graphs of the functions $y = a\sin(bx + c)$, $y = a\cos(bx + c)$ and $y = a\tan(bx + c)$ are related to the graphs of the functions $y = a\sin(x)$, $y = a\cos(x)$ and $y = a\tan(x)$ by a phase shift of $\frac{c}{b}$. If $\frac{c}{b}$ is positive, the shift is to the left. If $\frac{c}{b}$ is negative, it is to the right.





When $\frac{c}{b}$ is positive, everything happens $\frac{c}{b}$ units before it does on the related basic trigonometric graph.

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Example 4

Sketch the graph of $y = 3\sin\left(2\left(x - \frac{\pi}{2}\right)\right)$ for $0 \le x \le \pi$.

Solution

State the amplitude and period.

Amplitude = 3

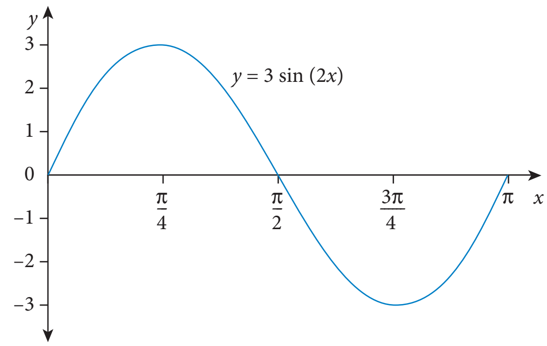
$\text{Period} = \frac{2\pi}{2} = \pi$

Using the formula $\frac{c}{b}$, find the horizontal translation.

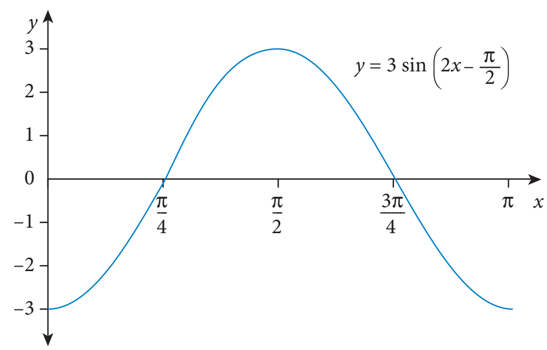
$\frac{c}{b} = -\frac{\pi}{2} \div 2 = -\frac{\pi}{4}$

Since the phase shift is negative, it is to the right.

Sketch the graph of $y = 3\sin(2x)$



Translate $y = 3\sin(2x)$ by $\frac{\pi}{4}$ to the right to sketch the graph of $y = 3\sin\left(2\left(x - \frac{\pi}{4}\right)\right)$.



For $\cos(bx + c)$ you can work the phase shift by asking 'When is $\cos(bx + c)$ the same as $\cos(x)$ for the zeros, maxima and minima?'. For $\cos(0)$, $bx + c = 0$ when $x = -\frac{c}{b}$. You can obviously use the same method with other trigonometric functions.

Example 5

Sketch the graph of $y = 2\cos\left(3x + \frac{\pi}{2}\right)$ for $0 \le x \le \pi$.

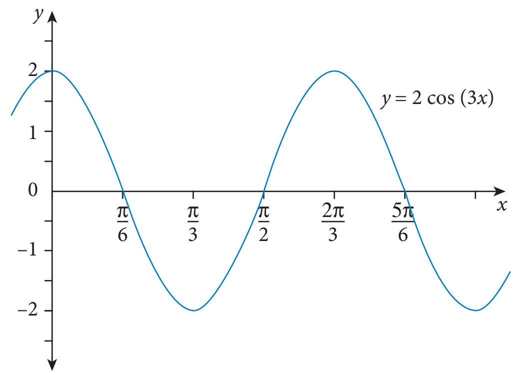
Solution

State the amplitude and period.

Amplitude = 3

Period = $\frac{2\pi}{3}$

Sketch the graph of $y = 2\cos(3x)$.



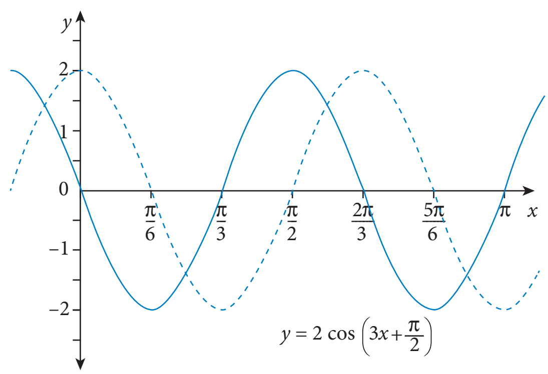
Find the horizontal translation for $y = 2\cos\left(3x + \frac{\pi}{2}\right)$.

$3x + \frac{\pi}{2} = 0$ gives $x = -\frac{\pi}{6}$ so everything moves $\frac{\pi}{6}$ left.

Translate $y = 3\sin(2x)$ by $\frac{\pi}{6}$ to the left to sketch the graph of

$y = 2\cos\left(3x + \frac{\pi}{2}\right) = 2\cos\left(3\left(x + \frac{\pi}{6}\right)\right)$

The zeros, maxima and minima all move back $\frac{\pi}{6}$.



Example **5** shows the technique of dotting in the graph of $y = 3\sin(2x)$ before applying the phase shift to draw the desired graph.

INVESTIGATION Musical beats

To tune a guitar, the frets are used to play a note on one string that should be the same as the base note of the next string. When the strings are in tune, the notes will be the same. When the strings are quite out of tune, they sound quite different. When they are close, but not quite the same, the sound will seem to beat (fade and strengthen rhythmically). This can be shown on a CAS calculator.

1 Start with the Main (Casio) or Calculator (TI) page. Consider a note that should have a frequency of 440 Hz. Define this as c. Define 442 as d.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Now go to the Graph page.

Put in $f1(x)$/Y1 = $\sin(2\pi cx)$, $f2(x)$/Y2 = $\sin(2\pi dx)$ and $f3(x)$/Y3 = $f1(x)$/Y1 + $f2(x)$/Y2, and turn $f1(x)$/Y1 and $f2(x)$/Y2 off.

Make the window settings or View Window so that $0 \le x \le 5$ and $-3 \le y \le 3$. Change d to 441 and graph it again.

Change d to 440.2, change the display so that $0 \le x \le 20$. Find what note corresponds to 440 Hz.

2 Try different values of c and d.

3 What do you find?

4 Write a short account of your results.

You can also sketch graphs with vertical translations as well as ones with varying amplitudes, periods and phase shifts.

WS
Sketching periodic functions – amplitude and period

WS
Sketching periodic functions

Example 6

Sketch the graph of $y = \frac{1}{3}\cos\left[4\left(x + \frac{\pi}{8}\right)\right] + 1$ for $0 \le x \le \pi$.

Solution

State the amplitude and period.

Amplitude $= \frac{1}{3}$

Period $= \frac{2\pi}{4} = \frac{\pi}{2}$

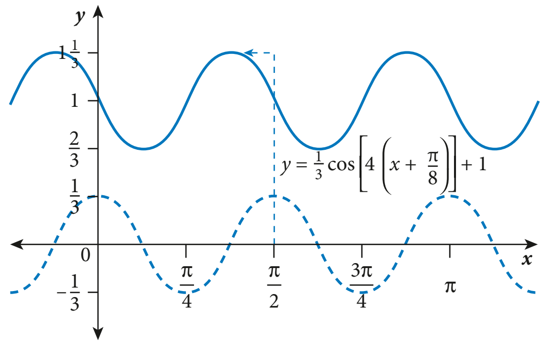
$y = \frac{1}{3}\cos\left[4\left(x + \frac{\pi}{8}\right)\right] + 1$ already has the phase shift worked out.

$4\left(x + \frac{\pi}{8}\right) = 0$ when $x = -\frac{\pi}{8}$, so everything moves $\frac{\pi}{8}$ left from $y = \frac{1}{3}\cos(4x) + 1$.

State the vertical translation.

The vertical translation is 1 unit up.

Sketch the graph of $\frac{1}{3}\cos(4x)$. Then sketch $y = \frac{1}{3}\cos\left[4\left(x + \frac{\pi}{8}\right)\right] + 1$ by moving everything $\frac{\pi}{8}$ left and 1 up.



WS
Sketching periodic functions – phase and vertical shift

IMPORTANT

The graphs of the functions $y = a\sin(bx + c) + d$ and $y = a\cos(bx + c) + d$ are related to the graphs of the functions $y = a\sin(x)$ and $y = a\cos(x)$ by as follows.

Summary of the features of sine and cosine functions

For the functions $y = a\sin(bx + c) + d = a\sin\left[b\left(x + \frac{c}{b}\right)\right] + d$

and $y = a\cos(bx + c) + d = a\cos\left[b\left(x + \frac{c}{b}\right)\right] + d$

- the amplitude is the size of a. If a is negative, the graph is upside down
- the period is $\frac{2\pi}{b}$
- the value $\frac{c}{b}$ is called the **phase shift** and is the horizontal translation
- the average (mean) value is d.

Sketching trigonometric graphs

Graphs of sine and cosine functions may be sketched from the equation by two methods.

Method 1: Identify the translations and changes of scale from the equation.

Method 2: Identify the:

- starting point
- zeros
- end of a cycle
- maxima and minima

Find the values of x that produce the sine or cosine of $0, \frac{\pi}{2}, \pi, \frac{\pi}{2}$ and 2π and the corresponding values of y.

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EXERCISE 12.02 General trigonometric functions

Concepts and techniques

1 Examples 4-6 Sketch graphs of each of the following and check with a CAS calculator, showing a full cycle of each.

a $y=4\sin\left[2\left(x-\frac{\pi}{3}\right)\right]+2$ b $y=2\cos\left[3\left(x+\frac{\pi}{6}\right)\right]-3$ c $y=5\sin\left[4\left(x+\frac{\pi}{8}\right)\right]-2$

2 Sketch graphs of each of the following and check with a CAS calculator, showing a full cycle of each.

a $y=2\sin\left(2x-\frac{\pi}{3}\right)+3$ b $y=3\cos(4x+\pi)-2$ c $y=4-3\sin\left(6x+\frac{3\pi}{2}\right)$

d $y=5\cos\left(2x-\frac{3\pi}{4}\right)-3$ e $y=3-5\cos\left(3x-\frac{\pi}{3}\right)$ f $y=4-3\sin\left(4x+\frac{\pi}{4}\right)$

Reasoning and communication

3 Sketch for $-2 \le x \le 2$

a $y = \sin(\pi x)$ b $y = 3\cos(2\pi x)$

4 In an unusual meteorological investigation, the temperature, T °C, in a town in central Victoria was found to fluctuate approximately according to the rule $T = 25 + 6\sin(0.1\pi t)$, where t is the number of hours after 10:00 a.m.

a Sketch a graph of the temperature fluctuations for a sufficient number of hours to be able to determine the maximum and minimum temperatures for that day and the next night.

b Use the graph to determine the maximum and minimum temperatures.

c When did they occur?

d At what time was the temperature:

i 27°C? ii 20°C?

e Why was this unusual?

5 In Calcutta, the highest mean monthly temperature is 29.45°C in June and the lowest is 18.3°C in December. Find a model for the temperature throughout the year and graph it.

12.03 APPROXIMATE SOLUTION OF TRIGONOMETRIC EQUATIONS

In previous sections you have sketched the graphs of the form $y = f(a(x-b)) + c$ where f is one of sine, cosine or tangent.

In this section you will use these graphs to solve trigonometric equations. In most real situations, exact answers are rare so you need to be able to find approximate answers.

Example 7

Sketch the graph of $y = \tan(2x) + 1$ for $0 \le x \le \pi$ and find solutions to the equation $\tan(2x) + 1 = 0$.

Solution

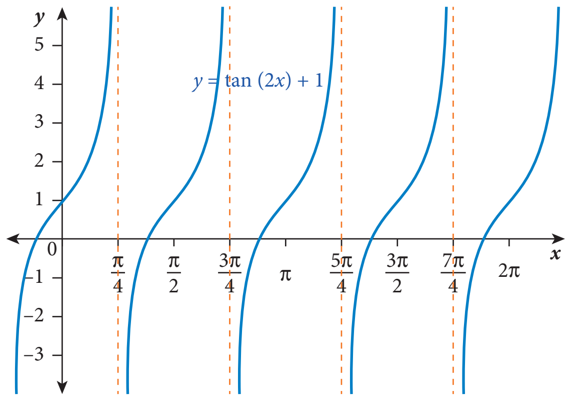
State the period.

Period $= \frac{\pi}{2}$

State the vertical translation.

The graph is translated 1 unit up compared to $y = \tan(2x)$.

Sketch the graph of $y = \tan(2x) + 1$

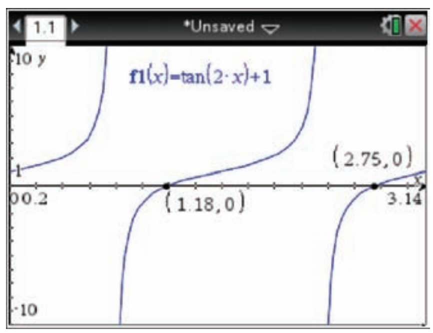


Find the solutions to the equation $\tan(2x) + 1 = 0$.

Solutions to the equation will be the x-intercepts of the graph. Approximate solutions to the equation $\tan(2x) + 1 = 0$ for $0 \le x \le \pi$ are $x = 1.2$, $x = 2.7$

TI-Nspire CAS

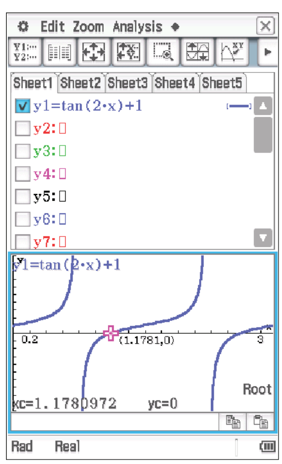
Use a Graph page and type in the function. with Window Settings of $0 \le x \le \pi$ and $-10 \le y \le 10$ (scales 0.2, 1). Use menu, 6: Analyse Graph and 1: Zero to find the zeros.



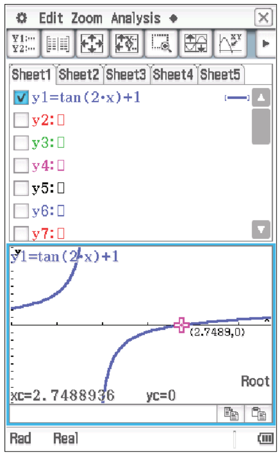
ClassPad

Use the **Graph** application, enter the function in ***y*1** and tap the box. Make **View Window** $0 \le x \le \pi$ and $-10 \le y \le 10$ (scales 0.2, 1). Tap to draw the graph.

Tap **Analysis**, **G-Solve** and **Root**.



You can obtain the larger root by resetting the lower boundary (such as by setting xmin=2) and repeating the commands given above.



You can find approximate solutions that correspond to the intersections of graphs.

Example 8

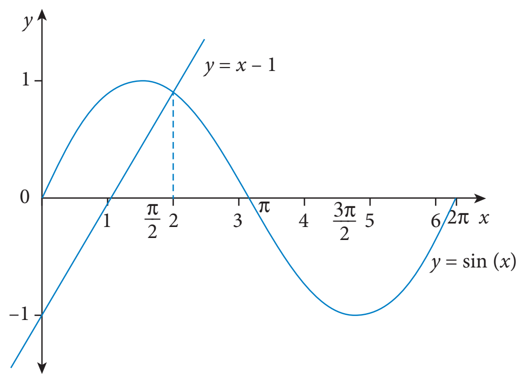
Find approximate solutions to the equation $\sin(x) = x - 1$ by sketching the graphs $y = \sin(x)$ and $y = x - 1$ on a Cartesian plane.

Solution

When sketching the two graphs together, we use $\pi \approx 3.14$, $2\pi \approx 6.28$ and so on to label the x-axis as shown.

To sketch $y = x - 1$, find the gradient and y-intercept or find the x- and y-intercepts.

x-intercept (where $y = 0$) is 1 and y-intercept (where $x = 0$) is -1.

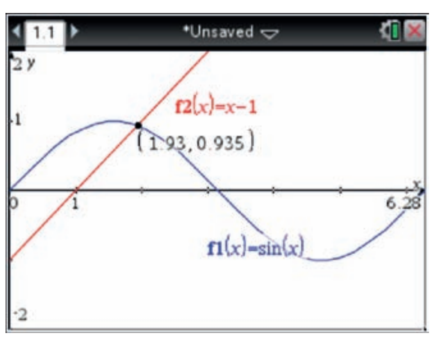


The solution to $\sin(x) = x - 1$ is at the point of intersection of the two graphs.

$x \approx 2$

TI-Nspire CAS

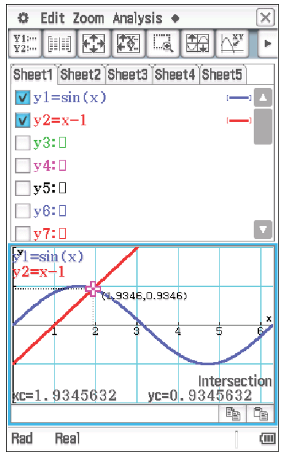
Use a Graph page and type in the function. Use Graph Entry/Edit to enter the second function. Set suitable values for the scales and use Analyse Graph to find the Intersection.



ClassPad

Use the **Graph** application and enter the functions in $y1$ and $y2$ and tap the boxes.

Under tap **View Window** to set $0 \le x \le 2\pi$ and $-2 \le y \le 2$ and tap to draw the graphs. Tap **Analysis**, **G-Solve** and **Intersection** to find the point.



You can often use graphs to find how many solutions an equation has.

Example 9

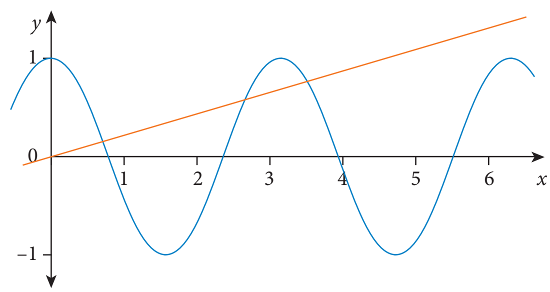
How many solutions are there for $\cos(2x) = \frac{x}{4}$ in the domain $0 \le x \le 2\pi$?

Solution

Sketch $y = \cos(2x)$ and $y = \frac{x}{4}$ on the same set of axes.

$y = \cos(2x)$ has amplitude 1 and period π.

$y = \frac{x}{4}$ has x-intercept 0 and passes through (4, 1).



There are 3 points of intersection of the graphs.

The equation $\cos(2x) = \frac{x}{4}$ has 3 solutions.

EXERCISE 12.03 Approximate solution of trigonometric equations

Concepts and techniques

1 Example 7 Sketch the graph of $y = 2\sin(x) - 1$ for $0 \le x \le 2\pi$ and hence find approximate solutions to the equation $2\sin(x) - 1 = 0$.

2 Example 8 Solve $\sin(x) = x$ for $0 \le x \le 2\pi$ graphically by sketching $y = \sin(x)$ and $y = x$ on the same number plane.

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3 Solve $\cos(x) = 2x - 3$ for $0 \le x \le 2\pi$ by finding the points of intersection of the graphs $y = \cos(x)$ and $y = 2x - 3$.

4 Solve $\tan(x) = x$ graphically in the domain $0 \le x \le 2\pi$.

5 Example 9 Show graphically that $\sin(x) = \frac{x}{2}$ has

a 2 solutions for $0 \le x \le 2\pi$

b 3 solutions for $-\pi \le x \le \pi$

Reasoning and communication

6 The depth of water in a harbour, in metres, can be modelled by $h = 2.8\cos(0.52t)$, where t is the number of hours since high tide. The depth is considered adequate for ships for only 2 hours either side of high tide. What is the minimum depth considered adequate?

12.04 EXACT SOLUTION OF TRIGONOMETRIC EQUATIONS

In this section you will solve trigonometric equations exactly. The basic technique for solving trigonometric equations exactly is to rearrange the original equation to obtain an equation of the type $f(a(x - b)) = c$, where f is the sin, cos or tan function. You then solve the $a(x - b) = f^{-1}(c)$, where f^{-1} gives the value(s) of x that give $\sin(x) = c$, $\cos(x) = c$ or $\tan(x) = c$.

Example 10

Solve the equation $2\sin\left(x - \frac{\pi}{4}\right) = -1$ for $0 \le x \le 2\pi$.

Solution

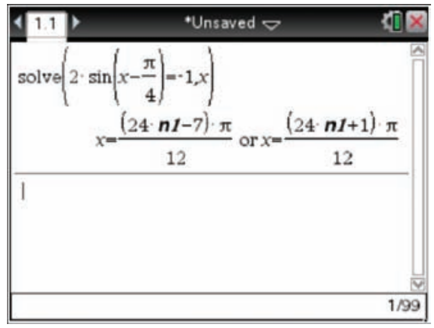
Write the equation.	$2\sin\left(x - \frac{\pi}{4}\right) = -1$
Rearrange to the form sin () =.	$\sin\left(x - \frac{\pi}{4}\right) = -\frac{1}{2}$
Find the value for which $\sin(x) = \frac{1}{2}$	$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$
Use the CAST diagram or another method to find the values for $\sin(x) = -\frac{1}{2}$	$\sin(x) = -\frac{1}{2}$ for $x = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \ldots$
Write and solve the first simplified equation.	$x - \frac{\pi}{4} = -\frac{5\pi}{6}$, so $x = -\frac{7\pi}{12}$
Do the next one.	$x - \frac{\pi}{4} = -\frac{\pi}{6}$, so $x = \frac{\pi}{12}$
Do the next one.	$x - \frac{\pi}{4} = \frac{7\pi}{6}$, so $x = \frac{17\pi}{12}$
Do the last one.	$x - \frac{\pi}{4} = \frac{11\pi}{6}$, so $x = \frac{25\pi}{12}$

But $-\frac{7\pi}{12}$ and $\frac{25\pi}{12}$ are outside $0 \le x \le 2\pi$.

The solutions are $x = \frac{\pi}{12}$ or $x = \frac{17\pi}{12}$.

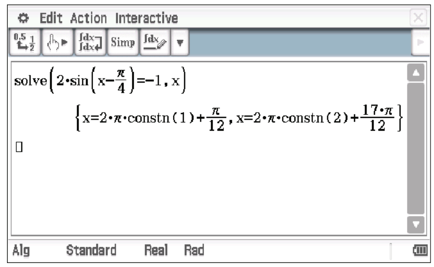
TI-Nspire CAS

Use a Calculator page, and make sure it is set on Radians and Auto or Exact Calculation mode.



ClassPad

Use the $\sqrt{\alpha}$ (Main) menu and make sure your calculator is set on radians (**Rad**) and **Standard** calculation.



Both calculators have integer constants in their answers. *n1* in the TI-Nspire means an integer, and constn(1) and constn(2) both mean integers for the CASIO. Only some integer values give answers with $0 \le x \le 2\pi$.

TI-Nspire CAS

Substitute $n1 = 1$ and $n1 = 0$.

$$x = \frac{(24 \times 1 - 7)\pi}{12} = \frac{17\pi}{12}$$

$$x = \frac{(24 \times 0 + 1)\pi}{12} = \frac{\pi}{12}$$

ClassPad

Substitute 0 for both constants.

$$x = 2\pi \times 0 + \frac{17\pi}{12} = \frac{17\pi}{12}$$

$$x = 2\pi \times 0 + \frac{\pi}{12} = \frac{\pi}{12}$$

In some trigonometric equations a **general solution** is required because the domain is not restricted and the answers go on and on.

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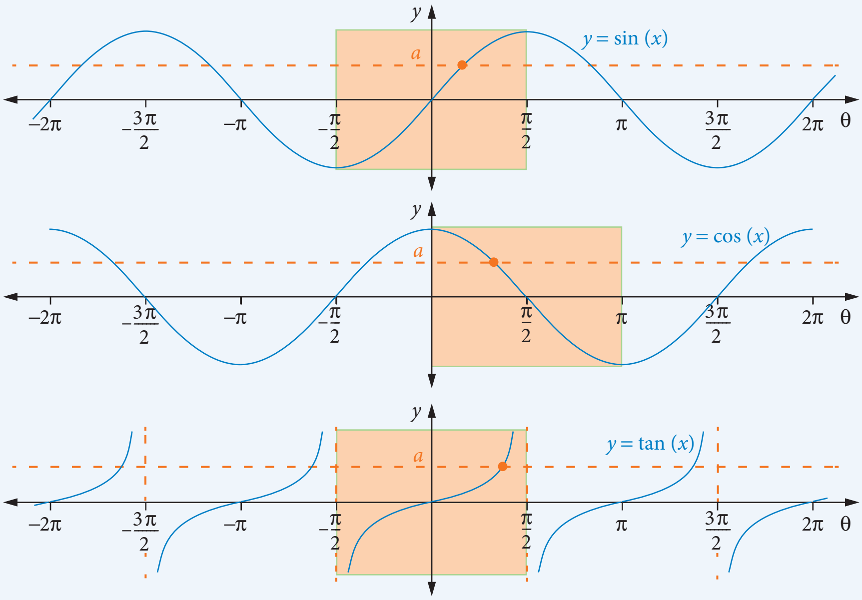
INVESTIGATION General solutions to trigonometric equations

You are familiar with the graphs of $\sin(x)$, $\cos(x)$ and $\tan(x)$ as shown here.



There is exactly one solution to equations of the form $\sin(x) = a$ in the range $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

There is exactly one solution to the equation $\tan(x) = a$ in the range $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

There is exactly one solution to the equation $\cos(x) = a$ in the range $0 \leq x \leq \pi$.

The functions $\arcsin(a)$, $\arccos(a)$ and $\arctan(a)$ are restricted to these ranges and the calculator functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ show only these values.

Notice that the range for arcsin and arctan are slightly different because tan is not defined for $x = \pm\frac{\pi}{2}$.

For the equation $\tan(x) = a$, if θ is the solution in the domain $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $x = n\pi + \theta$, where n is an integer. Remember that this means that n can be positive, zero or negative.

This type of solution is known as a general solution to the trigonometric equation. General solutions to $\sin(x) = a$ and $\cos(x) = a$ can also be expressed in terms of n and π, but the expressions are a little more complicated.

Work in groups of two or three to work out the general solutions of $\sin(\theta) = a$ and $\cos(\theta) = a$.

From the investigation above, you and others in your class may have found different ways of writing the general solutions of sine and cosine functions. To check whether they are the same, you could try working out the values for different cases. The formulas given below are one version of the way they can be written.

IMPORTANT

General solutions of trigonometric equations

For the equation $\sin(x) = a$, if θ is the solution in the domain $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then $x = n\pi + (-1)^n \times \theta$, where n is an integer.

For the equation $\cos(x) = a$, if θ is the solution in the domain $0 \le \theta \le \pi$, then $x = 2n\pi \pm \theta$, where n is an integer.

For the equation $\tan(x) = a$, if θ is the solution in the domain $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then $x = n\pi + \theta$, where n is an integer.

Example 11

Solve $\cos(x) = -\frac{\sqrt{2}}{2}$.

Solution

Find the value for which $\cos(x) = \frac{\sqrt{2}}{2}$.	$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$
Use the CAST diagram to find the solution in the range $0 \le \theta \le \pi$.	$\cos\left(\pi - \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ so $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$
Use symmetry, the graph of $\cos(x)$ or the formula.	The general solution is $x = 2n\pi \pm \frac{3\pi}{4}$ for $n \in \mathbf{Z}$.

Example **10** shows that your CAS calculator will find the general solution unless you restrict the range of solutions. You will usually be asked for answers within a restricted domain. In cases where the trigonometric function is not equal to one of the exact values, you can use your calculator to find approximate answers, or write the exact answer in terms of $\sin^{-1}(a)$, $\cos^{-1}(a)$ or $\tan^{-1}(a)$.

The values of $\sin^{-1}(a)$, $\cos^{-1}(a)$ and $\tan^{-1}(a)$ are always taken to be in the domains $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $0 \le \theta \le \pi$, and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ respectively. These values are sometimes called the **principal values**.

Example 12

Solve $2\tan\left(3x + \frac{\pi}{4}\right) - 7 = 0$ for $-\pi \le x \le 2\pi$, correct to 2 decimal places.

Solution

Write the equation.	$2\tan\left(3x + \frac{\pi}{4}\right) - 7 = 0$
Rearrange and simplify.	$\tan\left[3\left(x + \frac{\pi}{12}\right)\right] = 3.5$
Find the solution in $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.	$3\left(x + \frac{\pi}{12}\right) = \tan^{-1}(3.5) \approx 1.2925$
Write the general solution.	$3\left(x + \frac{\pi}{12}\right) = n\pi + \tan^{-1}(3.5)$

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Divide by 3. $x+\frac{\pi}{12}=\frac{n\pi+\tan^{-1}(3.5)}{3}$

Solve for x. $x=\frac{n\pi+\tan^{-1}(3.5)}{3}-\frac{\pi}{12}=\frac{(4n-1)\pi+4\tan^{-1}(3.5)}{12}$

Substitute values of n to find answers in the right range, from $-\pi\approx-3.14$ to $2\pi\approx6.28$.

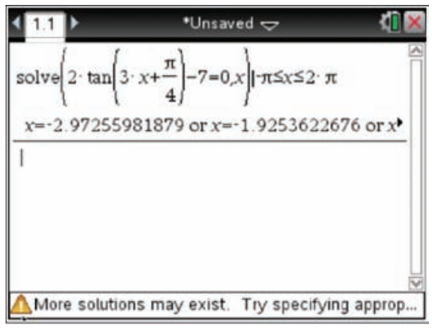
For $n=-4$, $x\approx-4.02$; for $n=-3$, $x\approx-2.97$; for $n=-2$, $x\approx-1.93$; for $n=5$, $x\approx5.41$; for $n=6$, $x\approx6.45$

Write the answers.

$x=-2.97, -1.93, -0.88, 0.17, 1.22, 2.26, 3.31, 4.36, 5.41$

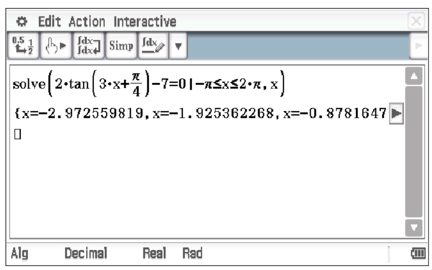
TI-Nspire CAS

Use solve(), but also specify a domain. The | and ≤ signs are in the menu obtained by pressing [ctrl] [=]. You can see the rest of the solutions using the arrows.



ClassPad

Use **solve()** with your calculator set on **Decimal**, but also specify a domain. The [|] and [≤] signs are in the menu obtained by pressing [Keyboard] then tapping [Math3]. You can see the rest of the solutions by tapping the [▶] or [◀].



EXERCISE 12.04 Exact solution of trigonometric equations

Concepts and techniques

1 Example 10 Solve the following equations for $0\le x\le 2\pi$.

a $2\cos\left(x+\frac{\pi}{2}\right)=-1$ b $-\sqrt{3}\tan\left(2x+\frac{\pi}{4}\right)=1$ c $\frac{1}{\sqrt{3}}\sin(3x)=-\frac{1}{2}$

2 Solve each of the following equations for the domain given.

a $2\cos(x)=-2 \quad -\pi\le x\le\pi$
b $\sin(x)-1=0 \quad 0\le x\le 2\pi$
c $1-\tan(x)=0 \quad -2\pi\le x\le 0$
d $6\cos(x)-3=3 \quad -\pi\le x\le 2\pi$
e $10\cos(x)+5=-5 \quad -\pi\le x\le\pi$
f $\cos(x)+8=8 \quad 0\le x\le 3\pi$
g $3\tan(x)+3=0 \quad -\pi\le x\le\pi$
h $2\sin(x)-1=0 \quad -\frac{\pi}{2}\le x\le\frac{3\pi}{2}$
i $6-4\cos(x)=4 \quad -\pi\le x\le 2\pi$
j $2\cos(x)-\sqrt{3}=0 \quad \pi\le x\le 3\pi$

3 Solve each of the following for $0° \le \theta \le 360°$.

a $\tan(\theta) = 1$ b $\sin(\theta) = \frac{1}{2}$ c $\cos(\theta) = -\frac{\sqrt{3}}{2}$ d $\sin(\theta) = -\frac{1}{\sqrt{2}}$

e $\cos(\theta) = 0$ f $\tan(\theta) = \sqrt{3}$ g $\cos(\theta) = -\frac{1}{2}$ h $\tan(\theta) = -\frac{1}{\sqrt{3}}$

i $\sin(\theta) = \sin(50°)$ j $\cos(\theta) = \cos(115°)$

4 Example 11 Solve the following equations, giving the general solution.

a $\sin(x) = -\frac{\sqrt{3}}{2}$ b $\cos(x) = \frac{\sqrt{3}}{2}$ c $\tan(x) = \frac{\sqrt{3}}{3}$

5 Example 12 Solve each of the following over the domain $0 \le x \le \pi$, correct to 4 decimal places.

a $3\sin(x) = 1$ b $4\cos(x) + 1 = 0$ c $5\tan(x) - 2 = 0$

d $9\cos(x) = -5$ e $7\sin(x) - 3 = 0$ f $3\tan(x) + 20 = 0$

6 Find all values of x in the domain $0 \le x \le 2\pi$ for which:

a $\sin(2x) = 0.5$ b $\cos(2x) = -1$ c $\sin(2x) = 1$

d $\sin(2x) = -0.5$ e $\sin(x) = \cos(x)$ f $\sin(x) = \sqrt{3}\cos(x)$

Reasoning and communication

7 Find general solutions for each of the following equations.

a $2\cos\left(2\left(x+\frac{\pi}{4}\right)\right)=\sqrt{3}$ b $2\sin\left(3\left(x+\frac{\pi}{3}\right)\right)+1=0$ c $\tan\left(2x-\frac{\pi}{6}\right)-1=0$

d $2\sin\left(2\left(x-\frac{\pi}{2}\right)\right)=-\sqrt{3}$ e $\cos\left(2x+\frac{\pi}{4}\right)-\frac{1}{\sqrt{2}}=0$

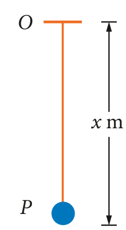
8 Find all solutions of the following equations for $-\pi \le x \le \pi$, correct to 2 decimal places,

a $3\sin\left(2\left(x-\frac{\pi}{3}\right)\right)=-2$ b $2\tan\left(3\left(x+\frac{\pi}{5}\right)\right)=3$ c $4\cos\left(4x+\frac{\pi}{4}\right)=3$

d $5\sin(3x-1)-4=0$ e $2\cos(5x+2)+1.3=0$

9 Find, correct to 2 decimal places, the two values of x closest to 0 for which $5\sin\left(2x-\frac{\pi}{6}\right)-2=0$.

10 A particle P bobs up and down on the end of an elastic string that is fixed at O. Its distance below O is given by the rule $x = 0.34 + 0.04\sin(4t)$ where x is in metres and t is the time in seconds since observation began.



a How far below O is the particle initially?

b How far below O is the particle at $t = 0.6$?

c What are the distance below O of the lowest point in the particle's path, and the value of t when the particle first reaches this point?

12.05 RECIPROCAL TRIGONOMETRIC FUNCTIONS

You have already done some work on the reciprocal trigonometric functions in Chapter 9.

IMPORTANT

Reciprocal functions

$$\operatorname{cosec}(x) = \frac{1}{\sin(x)}$$

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\cot(x) = \frac{1}{\tan(x)}$$

INVESTIGATION Important points in reciprocal functions

1. Use $\operatorname{cosec}(x) = \frac{1}{\sin(x)}$, $\sec(x) = \frac{1}{\cos(x)}$ and $\cot(x) = \frac{1}{\tan(x)}$ to complete the table below.

x	**0**	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
cosec (x)					
sec (x)					
cot (x)					

e.g.
$$\operatorname{cosec}\left(\frac{\pi}{2}\right) = \frac{1}{\sin\left(\frac{\pi}{2}\right)} = \frac{1}{1} = 1$$

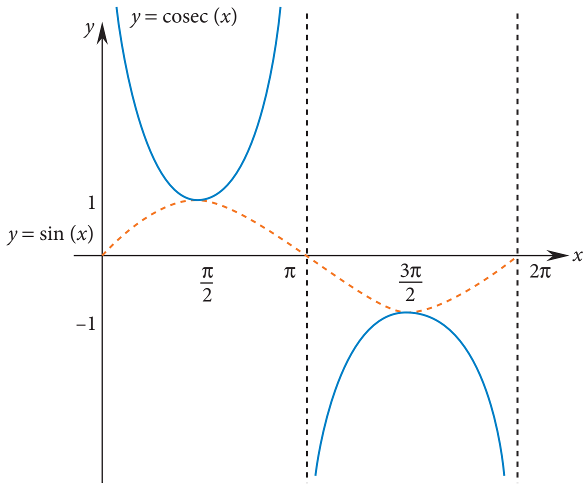
2. Find any asymptotes

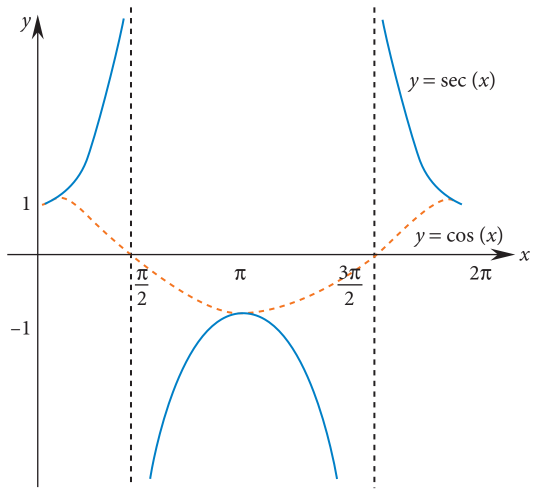
e.g.
$$\sec\left(\frac{\pi}{2}\right) = \frac{1}{\cos\left(\frac{\pi}{2}\right)} = \frac{1}{0} \quad (\text{undefined})$$

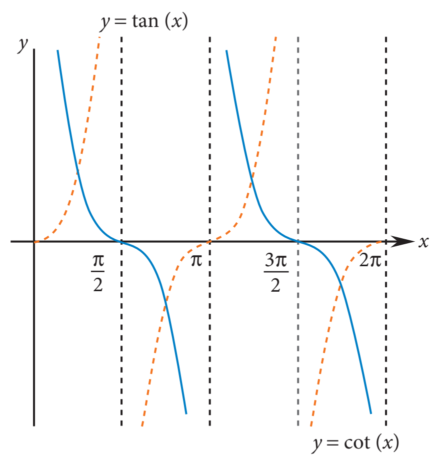
Discover what the values are on either side of the asymptotes.

3. Sketch each graph of the reciprocal trigonometric functions.

The graphs of the reciprocal ratios are drawn below.







Example 13

Sketch the graph of $y = \sin(2x)$ and hence sketch the graph of $y = \text{cosec}\,(2x)$, for $0 \le x \le \pi$.

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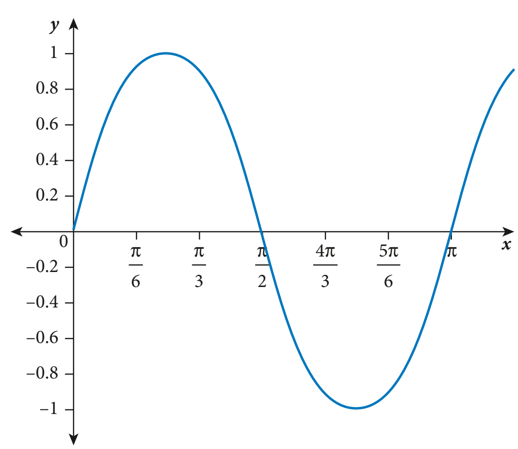
Sketch $y = \sin(2x)$ for $0 \le x \le \pi$.



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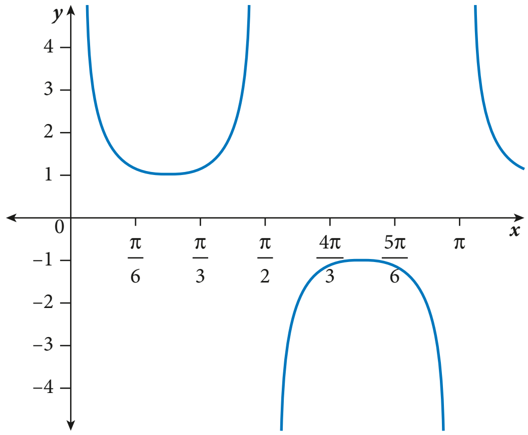
Sketch $y = \text{cosec}\,(2x)$ as the reciprocal of the graph of $y = \sin(2x)$.



Transformations of the reciprocal functions $y = \text{cosec}\,(x)$, $y = \sec\,(x)$ and $y = \cot\,(x)$ can be graphed using two methods.

IMPORTANT

Sketching reciprocal trigonometric function graphs

Method 1

Sketch the function for sin, cos or tan as appropriate and use the graph to draw the reciprocal function.

Method 2

Sketch the basic graph of $y = \text{cosec}\,(x)$, $y = \sec\,(x)$ and $y = \cot\,(x)$ and transform it using the properties of transformations.

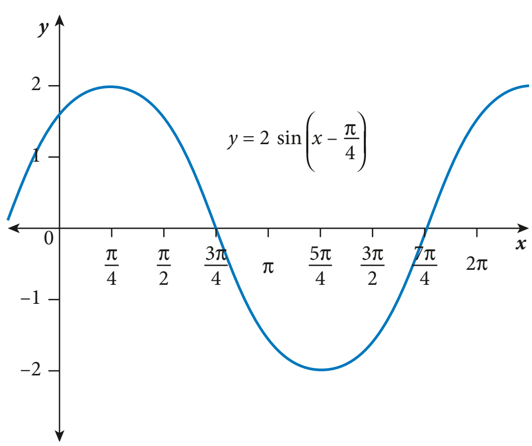
The following is an example showing Method 1.

Example 14

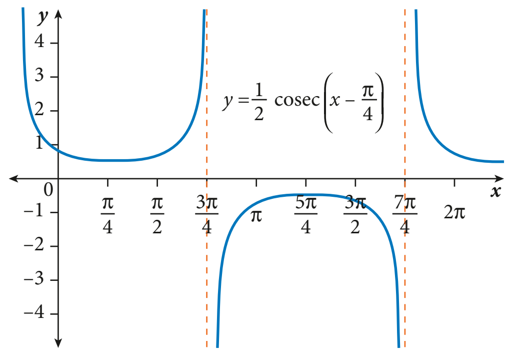
Sketch the graph of $y = 2\sin\left(x - \frac{\pi}{4}\right)$ and hence sketch the graph of $y = \frac{1}{2}\text{cosec}\left(x - \frac{\pi}{4}\right)$, for $0 \le x \le 2\pi$.

Solution

Sketch the graph of $y = 2\sin\left(x - \frac{\pi}{4}\right)$.



Sketch the graph of $y = \frac{1}{2}\text{cosec}\left(x - \frac{\pi}{4}\right)$, as the reciprocal of the graph of $y = 2\sin\left(x - \frac{\pi}{4}\right)$.



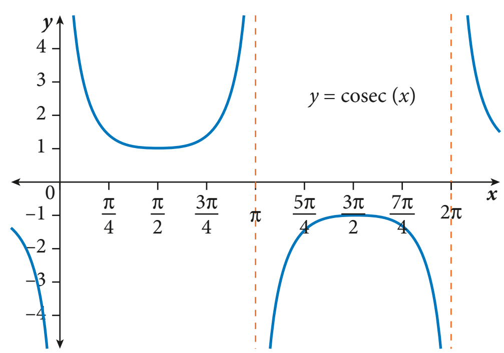
The following is an example showing Method 2.

Example 15

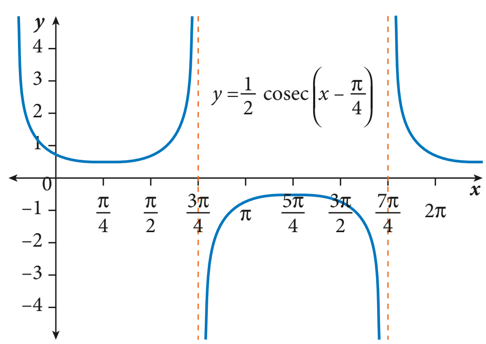
Sketch the graph of $y = \text{cosec}(x)$ and hence sketch the graph of $y = \frac{1}{2}\text{cosec}\left(x - \frac{\pi}{4}\right)$, for $0 \le x \le 2\pi$.

Solution

Sketch the graph of $y = \text{cosec}(x)$.



Sketch the graph of $y = \frac{1}{2}\text{cosec}\left(x - \frac{\pi}{4}\right)$, as the transformation of the graph of $y = \text{cosec}(x)$.



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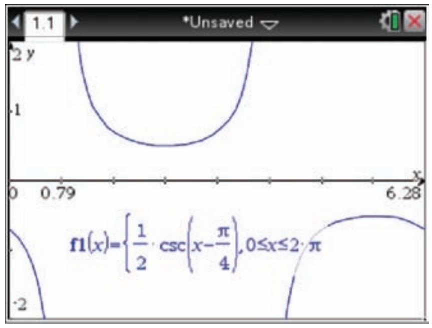
TI-Nspire CAS

Use a Graph page. The cosec function is shown as csc in the [trig] menu. Enter the function and the required range. Change the Window Settings to appropriate values.



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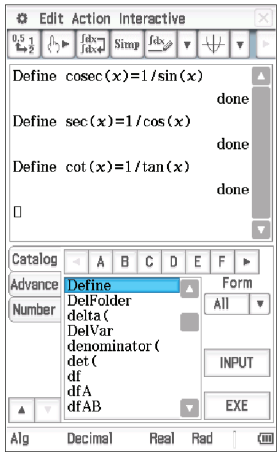
The reciprocal trigonometric functions are not listed on the calculator.

You must first define them, using the Main $\sqrt{\alpha}$ menu.

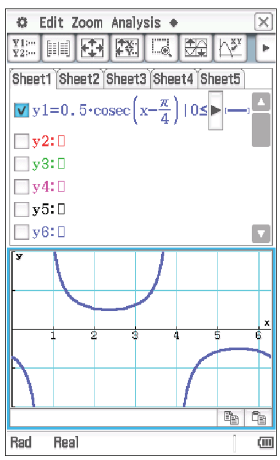
This need only be done once.



The graphs can then be sketched in the normal way.

$y1 = 0.5\text{cosec}\,(x - \frac{\pi}{4})\ |\ 0 \le x \le 2\pi$

Use **View Window** to set suitable values for x and y.



For the values $0 \le x \le 2\pi$, the y scale was chosen as -2 to 2 on the TI-Nspire to ensure that the graph was shown in approximately correct proportion.

EXERCISE 12.05 Reciprocal trigonometric functions

Concepts and techniques

1 Example 13 Use the graph of $y = \tan(x)$ to sketch $y = \cot(x)$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

2 Use the graph of $y = \cos(3x)$ to sketch the graph of $y = \sec(3x)$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

3 Use the graph of $y = \sin(4x)$ to sketch the graph of $y = \operatorname{cosec}(4x)$ from 0 to π.

4 Use the graph of $y = \tan(2x)$ to sketch the graph of $y = \cot(2x)$ from $-\pi$ to π.

5 Example 14 Use the graph of $y = \cos\left(x - \frac{\pi}{4}\right)$ to sketch the graph of $y = \sec\left(x - \frac{\pi}{4}\right)$ from $-\pi$ to π.

6 Use the graph of $y = \sin\left(x + \frac{\pi}{6}\right)$ to sketch the graph of $y = \operatorname{cosec}\left(x + \frac{\pi}{6}\right)$ from 0 to 2π.

7 Use the graph of $y = \tan\left(x + \frac{\pi}{4}\right)$ to sketch the graph of $y = \cot\left(x + \frac{\pi}{4}\right)$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

8 Example 15 Sketch the graph of $y = \operatorname{cosec}\left(x + \frac{\pi}{4}\right)$ from 0 to 2π.

9 Sketch the graph of $y = \sec\left(3x + \frac{\pi}{4}\right)$ from 0 to π.

10 Sketch the graph of $y = \cot\left(2x - \frac{\pi}{4}\right)$ from 0 to π.

11 Sketch the graph of $y = \sec\left(2x - \frac{\pi}{3}\right)$ from 0 to π.

12.06 *A* SIN (*X*) + *B* COS (*X*)

You can use the addition theorem for $\sin(x + y)$ to change expressions of the form $a\sin(x) + b\cos(x)$ to expressions of the form $r\sin(x + \alpha)$.

$\sin(x + \alpha) = \sin(x)\cos(\alpha) + \cos(x)\sin(\alpha)$

Multiplying by r gives
$r\sin(x + \alpha) = (r\cos(y))\sin(\alpha) +$
$(r\sin(\alpha))\cos(x)$

Choosing $a = r\cos(\alpha)$ and $b = r\sin(\alpha)$

Now $r = \sqrt{r^2 \times 1} = \sqrt{r^2 \sin^2(\alpha) + \cos^2(\alpha)} = \sqrt{r^2 \sin^2(\alpha) + r^2\cos^2(\alpha)} = \sqrt{a^2 + b^2}$

and $\tan(\alpha) = \dfrac{\sin(\alpha)}{\cos(\alpha)} = \dfrac{r\sin(\alpha)}{r\cos(\alpha)} = \dfrac{b}{a}$

IMPORTANT

$a\sin(x) + b\cos(x) = r\sin(x + \alpha)$ where $r = \sqrt{a^2 + b^2}$ and $\tan(\alpha) = \frac{b}{a}$

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Example 16

Write $\sqrt{3}\sin(x)+\cos(x)$ in the form $r\sin(x+\alpha)$.

Solution

Write the rule.

$a\sin(\theta)+b\cos(\theta)=r\sin(\theta+\alpha)$ where $r=\sqrt{a^2+b^2}$ and $\tan(\alpha)=\frac{b}{a}$

Write the values for a and b.

$a=\sqrt{3}, b=1$

Calculate r and α.

$$r=\sqrt{a^2+b^2}$$
$$=\sqrt{\sqrt{3}^2+1^2}$$
$$=\sqrt{3+1}$$
$$=\sqrt{4}$$
$$=2$$
$$\tan(\alpha)=\frac{b}{a}$$
$$=\frac{1}{\sqrt{3}}$$
$$\alpha=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$=30^\circ$$

Write the answer.

$\sqrt{3}\sin(x)+\cos(x)=2\sin(x+30^\circ)$

Example 17

Write $3\sin(\theta)+2\cos(\theta)$ in the form $r\sin(\theta+\alpha)$.

Solution

Write the rule.

$a\sin(\theta)+b\cos(\theta)=r\sin(\theta+\alpha)$ where $r=\sqrt{a^2+b^2}$ and $\tan(\alpha)=\frac{b}{a}$

Write the values for a and b.

$a=3, b=2$

Calculate r and α.

$$r=\sqrt{a^2+b^2}$$
$$=\sqrt{3^2+2^2}$$
$$=\sqrt{9+4}$$
$$=\sqrt{13}$$
$$\tan(\alpha)=\frac{b}{a}$$
$$=\frac{2}{3}$$
$$\alpha=\tan^{-1}\left(\frac{2}{3}\right)$$
$$=33^\circ 41'$$

Write the answer.

$3\sin(\theta)+2\cos(\theta)=\sqrt{13}\sin(\theta+33^\circ 41')$

Similar results can be found to create the expressions $r\cos(x \pm a)$ or $r\sin(x \pm a)$.

Example 18

Write the expression $3\cos(x) + 4\sin(x)$ in the form of $r\cos(x - \alpha)$.

Solution

Write the expansion of $r\cos(x - \alpha)$.	$r\cos(x - \alpha) = r\cos(x)\cos(\alpha) + r\sin(x)\sin(\alpha)$
Using trigonometric ratios, $\tan(\alpha) = \frac{b}{a}$.	$\tan(\alpha) = \frac{4}{3}$
Use $r = \sqrt{r^2\sin^2(\alpha) + r^2\cos^2(\alpha)}$	$r = \sqrt{b^2 + a^2} = \sqrt{4^2 + 3^2} = 5$
Write the answer.	$3\cos(x) + 4\sin(x) = 5\cos(x - \alpha)$, where $\alpha = \tan^{-1}\left(\frac{4}{3}\right)$

EXERCISE 12.06 $a\sin(x) + b\cos(x)$

Concepts and techniques

1 Examples 16, 17 Write each expression in the form $r\sin(\theta + \alpha)$.

a $2\sin(\theta) + \cos(\theta)$
b $\sin(\theta) + \sqrt{3}\cos(\theta)$
c $\sin(\theta) + \cos(\theta)$
d $5\sin(\theta) + 2\cos(\theta)$
e $4\sin(\theta) + \cos(\theta)$
f $3\sin(\theta) + \cos(\theta)$
g $2\sin(\theta) + 3\cos(\theta)$
h $4\sin(\theta) + 7\cos(\theta)$
i $5\sin(\theta) + 4\cos(\theta)$
j $3\sin(\theta) + 5\cos(\theta)$

2 Write each expression in the form $r\sin(\theta - \alpha)$.

a $\sin(\theta) - \cos(\theta)$
b $\sin(\theta) - 2\cos(\theta)$
c $\sin(\theta) - \sqrt{3}\cos(\theta)$
d $\sqrt{3}\sin(\theta) - \cos(\theta)$
e $5\sin(\theta) - 2\cos(\theta)$

3 Example 18 Write the expression $3\cos(\theta) + \sin(\theta)$ in the form $r\cos(\theta - \alpha)$.

4 Write the expression $\cos(\theta) - \sqrt{3}\sin(\theta)$ in the form $r\cos(\theta + \alpha)$.

5 Write the expression $9\sin(\theta) + 2\cos(\theta)$ in the form:

a $r\sin(\theta + \alpha)$
b $r\cos(\theta - \alpha)$

12.07 USING $A\cos(X) + B\sin(X)$

You can use expressions of the form $a\cos(x) + b\sin(x)$ to solve equations and sketch graphs.

In Example **18** you saw that $3\cos(x) + 4\sin(x)$ can be expressed as $5\cos(x - \alpha)$, where $\alpha = \tan^{-1}\left(\frac{4}{3}\right) \approx 53.13°$ correct to 2 decimal places.

Thus $3\cos(x) + 4\sin(x) \approx 5\cos(x - 53.13°)$.

Example 19

Use the fact that $3\cos(x) + 4\sin(x) = 5\cos(x - 53.13°)$ to graph the function $y = 3\cos(x) + 4\sin(x)$ for $0 \le x \le 360°$.

Solution

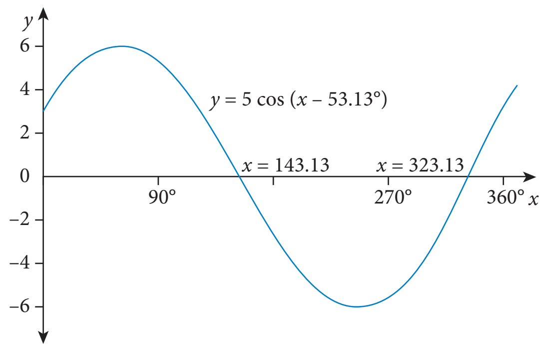
State the period, amplitude and phase shift of the graph of $y = 5\cos(x - 53.13°)$.

$P = \frac{360}{1} = 360°$

Amplitude = 5

Phase shift = 53.13° to the right.

Sketch the graph of $y = 5\cos(x - 53.13°)$.



You can make the equation, $3\cos(x) + 4\sin(x) = \text{constant}$, simpler to solve by using the equivalence $3\cos(x) + 4\sin(x) = 5\cos(x - 53.13°)$.

Example 20

Use the fact that $3\cos(x) + 4\sin(x) = 5\cos(x - 53.13°)$ to solve the equation $3\cos(x) + 4\sin(x) = 0$ for $0 \le x \le 360°$.

Solution

Simplify the equation $5\cos(x - 53.13°) = 0$.

$5\cos(x - 53.13°) = 0$

$\therefore \cos(x - 53.13°) = 0$

Solve the equation $5\cos(x - 53.13°) = 0$ for $0 \le x \le 360°$.

$\cos(x - 53.13°) = 0$

gives $(x - 53.13) = -90, 90, 270, 450, \ldots$

Consider the domain required.

$\therefore x = 90 + 53.13, 270 + 53.13, \ldots$

State solutions.

$x = 143.13°, 323.13°$

The solution in Example **20**, $x = 143.13°, 323.13°$, matches the zeros of the graph, $x = 143.13°, 323.13°$, in Example **19**.

Example 21

Solve $\sqrt{3}\sin(x) + \cos(x) = 1$ for $0° \le x \le 360°$.

Solution

Use the result for $a\sin(x) + b\cos(x)$. Find the values of a and b.

For $\sqrt{3}\sin(x)+\cos(x)$, $a=\sqrt{3}$ and $b=1$.

Calculate r and α.

$$r=\sqrt{a^2+b^2}$$
$$=\sqrt{\left(\sqrt{3}\right)^2+1^2}$$
$$=2$$
$$\tan(\alpha)=\frac{b}{a}$$
$$=\frac{1}{\sqrt{3}}$$
$$\therefore \alpha=30°$$

Rewrite the equation and the domain.

$\sqrt{3}\sin(x) + \cos(x) = 1$ for $0° \le x \le 360°$
$2\sin(x + 30°) = 1$ for $30° \le x + 30° \le 390°$

Simplify the equation.

$\sin(x+30°)=\frac{1}{2}$

Solve for $x + 30°$.

$x + 30° = 30°, 180° - 30°, 360° + 30°$
$x = 30°, 150°, 390°$

Write the answer.

$x = 0°, 120°, 360°$

EXERCISE 12.07 Using $a\cos(x) + b\sin(x)$

Concepts and techniques

1 Example 19 Express $3\cos(x) + 4\sin(x)$ in the form of $r\sin(x + \alpha)$. Hence graph the function $y = 3\cos(x) + 4\sin(x)$ for $0 \le x \le 360°$.

2 Express $4\sin(x) - 3\cos(x)$ in the form of $r\sin(x - \alpha)$. Hence graph the function $y = 4\sin(x) - 3\cos(x)$ for $0 \le x \le 360°$.

3 Express $2\cos(x) - 5\sin(x)$ in the form of $r\cos(x + \alpha)$. Hence graph the function $y = 2\cos(x) - 5\sin(x)$ for $0 \le x \le 360°$.

4 Examples 20, 21 Solve for $0° \le \theta \le 360°$

a $3\sin(\theta) + 4\cos(\theta) = 0$

b $5\cos(\theta) - 12\sin(\theta) = -3$

c $\sin(\theta) - \sqrt{3}\cos(\theta) = 0$

d $\sin(\theta) + \cos(\theta) = -1$

e $4\sin(\theta) - \cos(\theta) + 3 = 0$

f $\sin(\theta) - \cos(\theta) = 1$

g $\sqrt{2}\cos(\theta)+\sin(\theta)=1$

h $2\sin(\theta)-\cos(\theta)=\frac{\sqrt{5}}{2}$

i $3\cos(\theta) - 5\sin(\theta) + 2 = 0$

j $\sqrt{2}\cos(\theta) + \sin(\theta) + 1 = 0$

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Reasoning and communication

5 Find the general solution of $6\sin(\theta) - 8\cos(\theta) = 5$.

6 Find the general solution of $\sqrt{3}\sin(\theta) + \cos(\theta) = 1$

7 A radio wave follows the path of the equation $h = 9\sin\left(\frac{\pi t}{4}\right) + \cos\left(\frac{\pi t}{4}\right)$, where h (metres) is the height from a mean level and t (hours) is the time after 9 a.m.

a Simplify the expression $9\sin\left(\frac{\pi t}{4}\right) + \cos\left(\frac{\pi t}{4}\right)$, and show that it can be written in the form $r\sin\left(\frac{\pi t}{4} + \alpha\right)$.

b Using the expression found in part **a**, find the height of the radio waves at 9 a.m.

c Using the expression found in part **a**, find the height of the radio waves at 11 a.m.

d Find the time(s) in a 24-hour period when the height of the radio waves returns to that of 9 a.m.

12.08 MODELLING PERIODIC MOTION

The sine and cosine curves are used in many applications, including the study of waves. There are many different types of waves, including water, light and sound waves. Oscilloscopes display patterns of electrical waves on the screen of a cathode-ray tube.

Simple harmonic motion (such as the movement of a pendulum) is a wave-like or oscillatory motion when graphed against time. In 1581, when he was 17 years old, Galileo noticed a lamp swinging backwards and forwards in Pisa cathedral. He found that the lamp took the same time to swing to and fro, no matter how much weight it had on it. This led him to discover the pendulum.

IMPORTANT

A particle in **simple harmonic motion** (SHM) moves backwards and forwards (**oscillates**) in accordance with an equation of the type $x = a\cos(nt + \varepsilon)$, where a, n, ε are constants, x is its displacement from a central position and t is time. The central position is called the point of **equilibrium**.

A pendulum, a mass attached to a string, a buoy rising and falling with the tide, a violin or guitar string when plucked or a vibrating tuning fork are all examples of objects with simple harmonic motion (SHM).

Example 22

The displacement of a particle in metres over time t seconds is given by $x = 5\cos(2t)$. Describe and graph its motion.

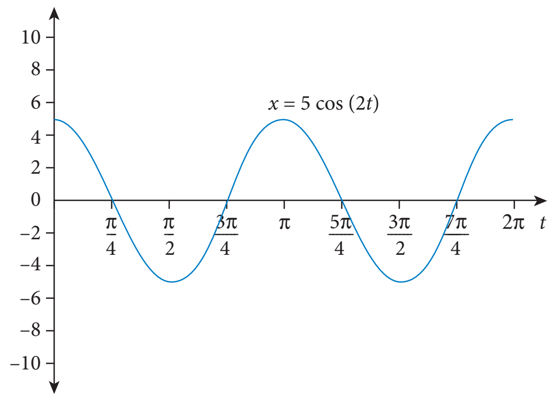
Solution

Find the period and amplitude.

Period $\frac{2\pi}{2} = \pi$

Amplitude $= 5$

Sketch the graph of $x = 5\cos(2t)$ for $0 \le t \le 2\pi$.



Describe the graph.

The particle starts at 5 metres from the origin. It then passes through the origin at $\frac{\pi}{4}$ seconds. The particle then travels to the other side of the origin until it is 5 metres in the other direction, and then back to the origin at $\frac{3\pi}{4}$ seconds. It then oscillates backwards and forwards.

At maximum displacement the particle is at rest.

The equation for SHM can also be in the form $x = a\sin(nt + \varepsilon)$.

The equation $x = a\cos(nt + \varepsilon)$ includes amplitude, period and also **phase shift**.

IMPORTANT

Since the position of a particle is given by $x = a\cos(nt + \varepsilon)$, the **amplitude** of the motion is a and the **period** is $\frac{2\pi}{n}$.

Example 23

A particle moves in a straight line with the equation of displacement given by $x = 3\sqrt{2}\cos\left(4t - \frac{\pi}{4}\right)$, where x is measured in metres and t in seconds. Describe and graph its motion.

Solution

State the period and amplitude.

$\text{Period} = \frac{2\pi}{4} = \frac{\pi}{2}$

$\text{Amplitude} = 3\sqrt{2}$

State the phase shift.

Express $x = 3\sqrt{2}\cos\left(4t - \frac{\pi}{4}\right)$ as

$x = 3\sqrt{2}\cos\left[4\left(t - \frac{\pi}{16}\right)\right]$.

Phase shift is $\frac{\pi}{16}$ in a horizontal positive direction.

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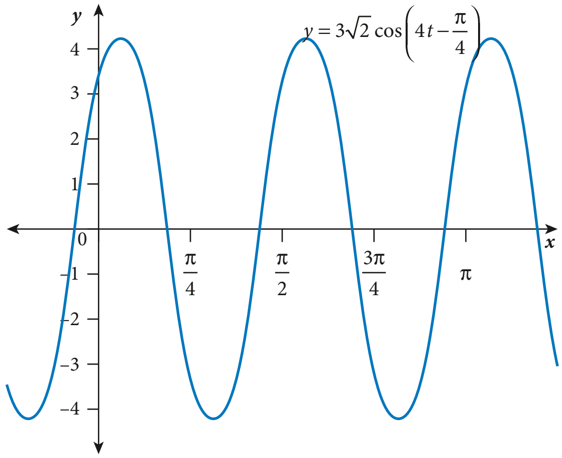
Sketch the graph of $x = 3\sqrt{2}\cos\left(4t-\frac{\pi}{4}\right)$ for two cycles.



Describe the graph.

The particle starts at 3 metres from the origin. It then moves further away until it reaches 4.24 metres from the origin. It turns around and passes through the origin at 0.59 seconds. The particle then travels to the other side of the origin until it is 4.24 metres in the other direction, and then back to the origin at 1.37 seconds. It then oscillates backwards and forwards.
At maximum displacement the particle is at rest.

Consider the displacement of a particle in cm given by $x = \sqrt{3}\cos(2t) + \sin(2t)$, where t is time in seconds.

You can use the relationship $a\sin(\theta) + b\cos(\theta) = r\sin(\theta + \alpha)$ where $r = \sqrt{a^2 + b^2}$ and $\tan(\alpha) = \frac{b}{a}$ to find the maximum distance from the equilibrium point.

Consider $x = \sqrt{3}\cos(2t) + \sin(2t)$

And

$$r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2}$$

$$\therefore r = 2$$

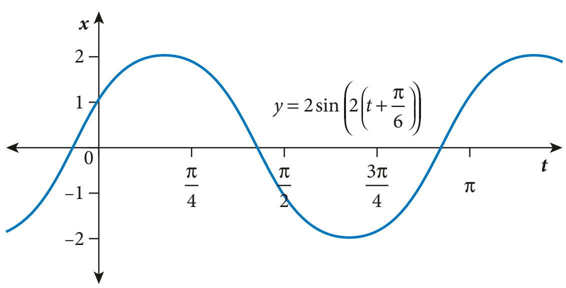
Using $\tan(\alpha) = \frac{b}{a}$

$$\tan(\alpha) = \frac{1}{\sqrt{3}}$$

$$\therefore \alpha = \frac{\pi}{6}$$

So $x = \sqrt{3}\cos(2t) + \sin(2t) = 2\sin\left(2\left(t + \frac{\pi}{6}\right)\right)$

A sketch of the graph is shown on the right.



From the graph it can be seen that the maximum distance from the origin or centre is 2 metres.

Other periodic motion can be modelled by a sine or cosine curve, giving applications of trigonometric graphs in real life situations, such as wave height.

Example 24

The table shows the highest average monthly temperatures in Sydney.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
°C	26.1	26.1	25.1	22.8	19.8	17.4	16.8	18.1	20.1	22.2	23.9	25.6

Bureau of Meteorology

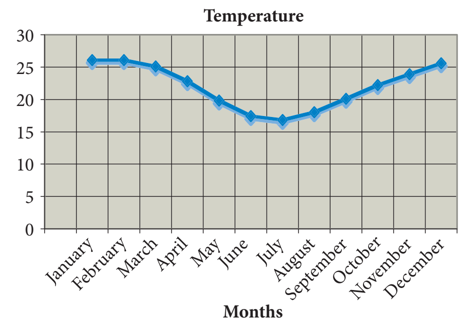
a Draw a graph of this data, by hand or on a calculator or computer.

b Is it periodic? Why would you expect it to be periodic?

c What is the period and amplitude?

Solution

a Draw the graph using any of the listed methods.



b Write the logical answer.

The graph looks like it is periodic, and we would expect it to be, since the temperature varies with the seasons. It goes up and down, and reaches a maximum in summer and a minimum in winter.

c This curve is approximately a cosine curve with one full period.
The maximum temperature is around 26° and the minimum is around 18°, so the centre of the graph is 22° with 4° on either side.

The period is 12 months.
The amplitude is 4.

EXERCISE 12.08 Modelling periodic motion

Reasoning and communication

1 Example 24 The graph shows the incidence of crimes committed over 24 years in Gotham City.

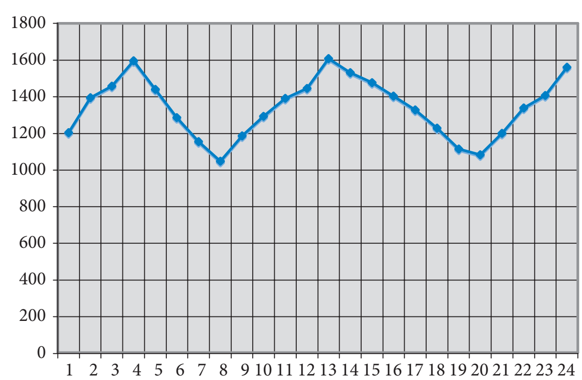
a Approximately how many crimes were committed in the 10th year?

b What was the

i highest and

ii lowest number of crimes?

c Find the amplitude and the period of the graph.



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2 Below is a table showing the average daylight hours over several months.

Month	Jan	Feb	Mar	Apr	May	June	July	Aug
Daylight hours	15.3	14.7	13.2	13.1	12.7	12.2	12.5	13.8

a Draw a graph to show this data.
b Is it periodic? If so, what is the period?
c Find the amplitude.

3 The table below shows the high and low tides over a three-day period.

Day	Friday				Saturday				Sunday			
Time	6.20 a.m.	11.55 a.m.	6.15 p.m.	11.48 p.m.	6.20 a.m.	11.55 a.m.	6.15 p.m.	11.48 p.m.	6.20 a.m.	11.55 a.m.	6.15 p.m.	11.48 p.m.
Tide m	3.2	1.1	3.4	1.3	3.2	1.2	3.5	1.1	3.4	1.2	3.5	1.3

a Sketch a graph showing the tides.
b Find the period and amplitude.
c Estimate the height of the tide at around 8 a.m. on Friday.

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4 Examples 22, 23 A particle is moving in simple harmonic motion, with displacement at any time t seconds given by $x = 2\cos(t)$. Sketch the graph of its displacement.

5 A particle is moving in SHM such that its displacement at any time t seconds is given by $x = 5\sin(t)$. Sketch the graph of its displacement.

6 A particle is oscillating about a central point so that its displacement at any time t seconds is given by $x = 4\cos(2t)$.
a Sketch the graph of its displacement.
b Find the times when the particle will have maximum displacement, and find this maximum displacement.

7 A particle's displacement is given by $x = 2\cos\left(t + \frac{\pi}{4}\right)$ m at time t seconds.
a Find the times at which the particle will be at the origin.
b Write down the period of the motion.
c Find the maximum displacement.

12 CHAPTER SUMMARY
TRIGONOMETRIC FUNCTIONS AND GRAPHS

- Equations of the form $y = \sin(n\theta)$ and $y = \cos(n\theta)$ have the **period** $P = \frac{2\pi}{n}$.
- Equations of the form $y = \tan(n\theta)$ have the **period** $P = \frac{\pi}{n}$.
- Equations of the form $y = a\sin(\theta)$ and $y = a\cos(\theta)$ have the amplitude a.
- Horizontal translations of the graphs $y = \sin(\theta)$, $y = \cos(\theta)$ and $y = \tan(\theta)$ to $y = \sin(\theta + c)$, $y = \cos(\theta + c)$ and $y = \tan(\theta + c)$ move the graph $-c$ units horizontally: called a **phase shift**.
- **Summary of the features of sine and cosine functions**

  For the functions
  $y = a\sin(bx + c) + d$

  and $y = a\cos(bx + c) + d$
  - the amplitude is the magnitude of a
  - the period is $\frac{2\pi}{b}$
  - the value $\frac{c}{b}$ is called the **phase shift** and is the horizontal translation
  - the average (mean) value is d.
- Reciprocal trigonometric functions are found by reciprocating the trigonometric functions of sine, cosine and tangent.
- Reciprocal trigonometric relationships are $\operatorname{cosec}(x) = \frac{1}{\sin(x)}$, $\sec(x) = \frac{1}{\cos(x)}$, $\cot(x) = \frac{1}{\tan(x)}$.
- Graph reciprocal trigonometric equations by doing one of the following.

  1 Sketch a sine, cosine, or tangent graph and reciprocate.

  2 Sketch the basic $y = \operatorname{cosec}(x)$, $y = \sec(x)$, $y = \cot(x)$ and transform.
- For the expression $a\sin(x) + b\cos(x) = r\sin(x + \alpha)$, where $r = \sqrt{a^2 + b^2}$ and $\tan(\alpha) = \frac{b}{a}$
  - convert sums $a\cos(x) + b\sin(x)$ to $R\cos(x \pm a)$ or $R\sin(x \pm a)$ and apply these to sketch graphs
  - solve equations of the form $a\cos(x) + b\sin(x) = c$ and solve problems
  - model periodic motion using sine and cosine functions and understand the relevance of the period and amplitude of these functions in the model
- Applications of trigonometric functions to model other periodic phenomena.

CHAPTER REVIEW 12

TRIGONOMETRIC FUNCTIONS AND GRAPHS

Multiple choice

1 Example 1 The period of the graph $y = -3\sin(4x)$ is:

A $-\frac{\pi}{2}$ B $\frac{\pi}{2}$ C π D $\frac{3\pi}{2}$ E 2π

2 Example 2 The amplitude of the graph $y = -3\sin(4x)$ is:

A -3 B 1 C 2 D 3 E 4

3 Example 3 The amplitude, period and phase shift of the graph $y = 3\sin\left(\theta - \frac{\pi}{4}\right)$ are respectively:

A $3, \pi, \frac{\pi}{4}$ horizontally to the left

B $-3, 2\pi, \frac{\pi}{4}$ horizontally to the right

C $\pi, 3, \frac{\pi}{4}$ horizontally to the left

D $3, 2\pi, \frac{\pi}{4}$ horizontally to the right

E $3, 2\pi, \frac{\pi}{4}$ horizontally to the left

4 Examples 4-6 The period and phase shift of the graph $y = \sin(3x + \pi)$ are respectively:

A $3\pi, \frac{\pi}{3}$ horizontally to the left

B $\frac{2\pi}{3}, \frac{\pi}{3}$ horizontally to the right

C $\frac{2\pi}{3}, \frac{\pi}{3}$ horizontally to the left

D $2\pi, \frac{\pi}{3}$ horizontally to the right

E $2\pi, \frac{\pi}{3}$ horizontally to the left

5 Examples 16-18 When $3\cos(x) + 3\sin(y)$ is written in the form $r\cos(x - \alpha)$,

A $r = 3$ and $\alpha = \frac{\pi}{4}$

B $r = 3\sqrt{2}$ and $\alpha = \frac{3\pi}{4}$

C $r = 3$ and $\alpha = -\frac{\pi}{4}$

D $r = 3\sqrt{2}$ and $\alpha = \frac{\pi}{4}$

E $r = \sqrt{6}$ and $\alpha = \frac{3\pi}{4}$

Short answer

6 Examples 4-6 Find the significant points of the cycle and hence sketch about one cycle of the graph of each of the following.

a $y = 3\cos\left(2x - \frac{\pi}{6}\right) + 4$ b $y = 3\cos\left(3x - \frac{\pi}{4}\right) - 2$ c $y = -2 - 2\sin\left(2x - \frac{\pi}{3}\right)$

7 Examples 4-6 CAS Draw each of the following.

a $y = 5\sin\left(2x + \frac{\pi}{4}\right) - 4$ b $y = 4\cos\left(3x + \frac{\pi}{6}\right) - 5$

c $y = 4\cos(x) + 2\cos(2x) + \cos(4x)$

8 Examples 8, 9 Solve $\sin(2x) = x$ for $0 \le x \le 2\pi$ by graphical means.

9 Example 8 Draw the graphs of $y = \sin(x)$ and $y = \cos(x)$ for $0 \le x \le 2\pi$ on the same set of axes. Use your graphs to solve the equation $\sin(x) = \cos(x)$ for $0 \le x \le 2\pi$

10 Example 10 CAS Solve the equation $-2\cos\left(2x+\frac{\pi}{6}\right)=1$ for $0 \le x \le 2\pi$.

11 Examples 11, 12 Solve the following for $\pi \le \alpha \le 2\pi$.

a $\sin(\alpha) = 0$ b $2\cos(\alpha) = 1$ c $\tan(\alpha) - 1 = 0$ d $1 + \tan(\alpha) = 1$

e $2\cos(\alpha) = \sqrt{3}$ f $\sqrt{3}\tan(\alpha) + 1 = 0$ g $\sqrt{2}\cos(\alpha) + 1 = 0$ h $\tan(\alpha) - \sqrt{3} = 0$

i $\cos(\alpha) = \cos\left(\frac{\pi}{8}\right)$ j $\sin(\alpha) = \sin\left(\frac{\pi}{6}\right)$

12 Examples 13-15 Sketch the following graphs from 0 to 2π.

a $y = \sec\left(x+\frac{\pi}{6}\right)$ b $y = \mathrm{cosec}\left(2x+\frac{\pi}{4}\right)$

13 Examples 13-15 Use the definitions to find the following.

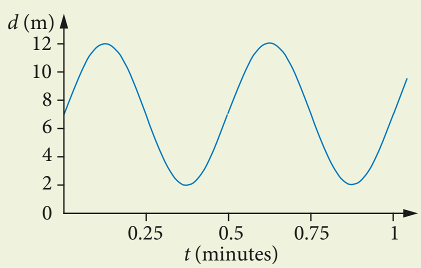
a $\cot\left(\frac{\pi}{3}\right)$ b $\mathrm{cosec}\left(\frac{3\pi}{4}\right)$ c $\sec\left(-\frac{11\pi}{6}\right)$ d $\mathrm{cosec}\left(\frac{4\pi}{3}\right)$

14 Examples 16-18 Write the expression $4\cos(x) + 3\sin(x)$ in the form $r\cos(x - \alpha)$.

Application

15 By expressing $6\cos(x) + \sin(x)$ in the form $r\cos(x - \alpha)$, solve the equation $6\cos(x) + \sin(x) = 1$ for $0 \le x \le 360°$.

16 Juanita is lying on a tropical beach, enjoying the sound of the waves. She has just finished her exams and so has plenty of time to observe the movement of the waves. She notices that the waves appear to roll up the beach at regular time intervals, and she is able to estimate the distance of the wave front from her toes over time. Idly, she scratches Cartesian axes in the sand and sketches the distance of the wave front from her toes against time in minutes. At this point she realises that the distance can be modelled by a sine curve $d = a\sin(bt) + c$



where a, b and c are positive constants.

a State the maximum and minimum distances of the waves from her feet.

b How many waves wash up on the beach each hour?

c Find the values of a, b and c.

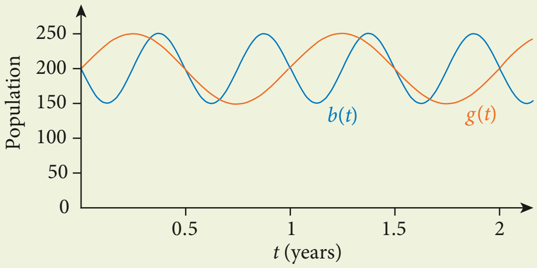
Her beach umbrella is stuck into the sand 4.5 m closer to the wave front than her toes and the waves are washing over its base.

d Calculate the percentage of time for which the base of the umbrella is in the water.

17 In a large butterfly enclosure the populations of two species are given by:

Orange: $g(t) = a + b \sin(mt)$

Blue: $b(t) = c - d \sin(nt)$



where t gives the number of years from when the observations began and a, b, c, d, m and n are positive constants.

The graphs of $g(t)$ and $b(t)$ for the first 2 years are shown.

Practice quiz

a State the values of a, b and m and hence write the rule for $g(t)$.

b State the values of c, d and n and hence write the rule for $b(t)$.

c Graphically determine the times when the populations are equal and state the populations at these times.

d Find the number of *days* after which the population of blues is first less than 180.